Around the Schwarz lemma

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the Schwarz lemma

the automorphism group of the unit disc

Ahlfors' version of the Schwarz lemma

applications

the Schwarz lemma

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Let $B(0,r) = \{z \in \mathbb{C} : |z| < r\}$ be the open disc of radius r and $B[0,r]$ be closed disc of radius r.

Schwarz Lemma: Suppose that $f : B(0,1) \to B(0,1)$ is a holomorphic function with $f(0) = 0$. Then

(i) $|f(z)| \leq |z|, z \in B(0,1)$

(ii) $|f'(0)| \leq 1$

(iii) If there exists $z_0 \in B(0,1), z_0 \neq 0$ such that $|f(z_0)| = |z_0|$ or $|f'(0)| = 1$, then f must be of the form $f(z) = cz, z \in B(0,1)$, for some c with $|c| = 1$.

The Proof of the Schwarz lemma is an in immediate Consequence of the Maximum modulus principle.

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The maximun modulus principle: Suppose that Ω is a bounded domain (open and connected) in $\mathbb{C}$, f is a holomorplic function on Ω, and $B[a,r] \subseteq \Omega$. Then

$$|f(a)| \leqslant \sup\{|f(a+re^{i\theta})| : \theta \in \mathbb{R}\}.$$

Equality occurs if and only of f is constant.

Proof: Assume that $|f(a+re^{1\theta})| \leqslant |f(a)|$, $\theta \in \mathbb{R}$. Then the holomorplic function f has a power series expansion

$$f(z) = \sum c_n (z-a)^n, \quad z \in B(a,R),$$

and If $0 < r < R$, we have (Parsevel's formula)

$$\sum |c_n|^2 r^{2n} = 1/2\pi \int_{-\pi}^{\pi} \left| f\left(a + re^{i\theta}\right)\right|^2 d\theta.$$

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It follows that

$$\sum |\, c_n \,|^2\, r^{2n} \leqslant |f(a)|^2 = |c_0|^2.$$

Hence $0 = c_1 = c_2 = \cdots,$ implying $f(z) = f(a),\ z \in B(a,r).$ Since Ω is connected, it follows f must be a constant.

Proof of the Schwarz Lemma: Since $f(0) = 0,$ we have $a_0 = 0,\ f(z) = \sum_{n=1}^{\infty} a_n z^n,\ z \in B(0,1).$ Let $h(z) = \sum_{n=1}^{\infty} a_n z^{n-1},$ $z \in B(0,1).$ Then h is holomorphic on $B(0,1)$ and $f(z) = zh(z),\ z \in B(0,1).$ By the maximum modulus theorem

$$\sup\{|h(z)| : |z| \leq r\} = \sup\{|h(z)| : |z| = r\} = \frac{1}{r}\sup\{|f(z)| : |z| \leq r\}$$

for all $r,\ 0 < r < 1.$ Since $|f(z)| \leq 1$ for all $z \in B(0,1)$ we get, on letting $r \to 1,$ that $\sup\{|h(z)|; z \in B(0,1)\} \leq 1.$ Hence $|f(z)| \leq |z|$ for all $z \in B(0,1).$ This completes the proof of (i).

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Moreover, if $|f(z_0)| = |z_0|$ for some $z_0 \neq 0$, then $|h(z_0)| = 1$ and, by the maximum modulus theorem, h is a constant function of modulus 1, that is, there exists $\theta \in \mathbb{R}$ such that $f(z) = zh(z) = e^{i\theta} z$, $z \in B(0,1)$.

Since $\frac{f(z)}{z} = h(z)$ for $z \in B(0,1)\setminus\{0\}$ and $f(0) = 0$ it follows that

$$|f'(0)| = \lim_{\substack{z \to 0 \\ z \neq 0}} \frac{|f(z)|}{|z|} = \lim_{z \to 0} |h(z)| = |h(0)| \leq 1.$$

If $|f'(0)| = |h(0)| = 1$, then by the maximum modulus theorem, h is a constant function of modulus 1 and as before $f(z) = e^{i\theta} z$, $z \in B(0,1)$.

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Theorem: For a fixed $\alpha \in B(0,1)$, $\varphi_\alpha(z) := \frac{z-\alpha}{1-\bar{\alpha}z}$ is a rational function mapping $B(0,1)$ onto $B(0,1)$ and also $\partial B(0,1)$ onto $\partial B(0,1)$. It is one to one on $B[0,1]$. The inverse of φ_α is $\varphi_{-\alpha}$.

Proof: The function φ_α is holomorphic in the whole plane except for $z = 1/\bar{\alpha}$ which is outside $B[0,1]$. We see that $\varphi_{-\alpha}(\varphi_\alpha(z)) = z$. Thus φ_α is one-one and $\varphi_{-\alpha}$ is its inverse. If $t \in \mathbb{R}$, then

$$\left|\frac{e^{it}-\alpha}{1-\bar{\alpha}e^{it}}\right| = \left|\frac{e^{it}-\alpha}{e^{-it}-\bar{\alpha}}\right| = 1,$$

and we see that φ_α maps $\partial B(0,1)$ into itself. The same is true of $\varphi_{-\alpha}$ hence $\varphi_\alpha(\partial B[0,1]) = \partial B[0,1]$. Applying the maximum modulus principle, we conclude that $\varphi_\alpha(B(0,1)) \subseteq B(0,1)$. This is equally true of $\varphi_{-\alpha}$.

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Schwarz lemma, in general

Suppose that α, β are complex numbers; $|\alpha|, |\beta| < 1$
Question: How large can $|f'(\alpha)|$ be if $f: B(0,1) \to B(0,1)$ and $f(\alpha) = \beta$?

Answer: $|f'(\alpha)| \le \frac{1-|\beta|^2}{1-|\alpha|^2}$. To verify this, put $g = \varphi_\beta \circ f \circ \varphi_{-\alpha}$. Since $\varphi_\beta,\ \varphi_\alpha : B(0,1) \to B(0,1)$, it follows that $g : B(0,1) \to B(0,1)$. Also, $g(0) = 0$. Thus $|g'(0)| \le 1$ by the Schwarz lemma. Differentiating g using the chain rule, we have

$$g'(0) = \varphi'_\beta(\beta)\ \ f'(\alpha)\ \ \varphi'_{-\alpha}(0).$$

This verifies the correctness of our answer since

$$\varphi'_\alpha(0) = 1 - |\alpha|^2,\ \ \varphi'_\alpha(\alpha) = (1-|\alpha|^2)^{-1}.$$

Equality occurs if and only if $g(z) = cz$, for some $c : |c| = 1$. Thus $f(z) = \varphi_{-\beta}(c\varphi_\alpha(z)),\ \ z \in B(0,1)$.

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the automorphism group of the unit disc

A remarkable feature: f is a rational function, although no continuity assumption was made on f near the boundary.

Theorm: Suppose that f is a bijective holomorphic function on $B(0,1)$ and that $f(\alpha)=0$. Then there exists a constant $c:|c|=1$ such that

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distance decreasing

Let $f : B(0,1) \to B(0,1)$ be holomorphic. Then for any $a, b \in B(0,1)$, $\left|\frac{f(a)-f(b)}{1-\overline{f(a)}f(b)}\right| \leq \left|\frac{a-b}{1-\bar{a}b}\right|$.

In particular, $\frac{|f'(z)|}{1-|f(z)|^2} \leq \frac{1}{1-|z|^2}$ for all $z \in B(0,1)$.

Riemannian Metric: A C^2 function $\varphi : \Omega \to \mathbb{R}_+$ defined on an open connected subset Ω of $\mathbb{C}$, is said to be a Riemannian metric. If $f : B(0,1) \to B(0,1)$ is holomorphic, then it is distance decreasing with respect to the Poincare metric: $\rho(z) := \frac{1}{1-|z|^2}$ defined on $B(0,1)$, that is, $f^*(\rho) \leq \rho$.

Here for any metric $\varphi : \Omega \to \mathbb{R}_+$ on Ω, and any C^2 -function $f : \tilde{\Omega} \to \Omega$, the pull-back $f^*\varphi$ is

$$(f^*\varphi)(z) := |f'(z)|\,\varphi(f(z))$$

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Ahlfors' version of the Schwarz lemma

Ahlfors' Schwarz lemma

The Gaussian curvature of a Riemannian metric φ^2 is defined to be

$$K_\varphi(z) = -\varphi(z)^{-2}\ \Delta \log \varphi(z).$$

Claim: We will verify that any holomorphic function $f : \Omega \to B(0,r)$ defines a metric $f^* p_r$ of constant negative curvature on $\Omega \setminus \{f' = 0\}$, where $p_r(z) := \frac{r}{r^2 - |z|^2}$ is the Poincare metric of $B(0,r)$ and

$$(f^* p_r)(z) := \frac{r|f'(z)|}{r^2 - |f(z)|^2},\ z \in \Omega \setminus \{f' = 0\}.$$

Ahlfors' Lemma: Let $\varphi \geq 0$ be a continuous function on $B(0,1)$. Assume that φ is C^2 on the open set $D_\varphi := \{\varphi > 0\}$. Suppose $K_\varphi \leq -\eta$, on D_φ for some $\eta > 0$. Then

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Proof of the claim: For a holomorphic function f defined on an open connected set $\Omega \subseteq \mathbb{C}$ with $|f(z)| \le r$, we have

$$\begin{aligned}
\Delta \log(r^2 - |f|^2)^{-1} &= -4\frac{\partial^2}{\partial \bar{z}\partial z}\log(r^2-|f|^2)\\
&= 4\frac{\partial}{\partial \bar{z}}\left(\frac{\bar{f}f'}{r^2-|f|^2}\right)\\
&= 4f'\left(\frac{\bar{f}'}{r^2-|f|^2}+\frac{\bar{f}f\bar{f}'}{(r^2-|f|^2)^2}\right)\\
&= 4|f'|^2\left(\frac{r^2-|f|^2+|f|^2}{(r^2-|f|^2)^2}\right)\\
&= 4\left(\frac{r|f'|}{r^2-|f|^2}\right)^2
\end{aligned}$$

In particular,

$$\Delta \log \frac{r}{r^2-|z|^2} = 4\left(\frac{r}{r^2-|z|^2}\right)^2, \ |z|<r.$$

Proof of the claim: For a holomorphic function f defined on an open connected set $\Omega \subseteq \mathbb{C}$ with $|f(z)| \leq r$, we have

$$\begin{aligned}
\Delta \log(r^2 - |f|^2)^{-1} &= -4 \frac{\partial^2}{\partial \bar{z} \partial z} \log(r^2 - |f|^2) \\
&= 4 \frac{\partial}{\partial \bar{z}} \left(\frac{\bar{f} f'}{r^2 - |f|^2} \right) \\
&= 4 f' \left(\frac{\bar{f}'}{r^2 - |f|^2} + \frac{\bar{f} f \bar{f}'}{(r^2 - |f|^2)^2} \right) \\
&= 4 |f'|^2 \left(\frac{r^2 - |f|^2 + |f|^2}{(r^2 - |f|^2)^2} \right) \\
&= 4 \left(\frac{r |f'|}{r^2 - |f|^2} \right)^2
\end{aligned}$$

In particular,

$$\Delta \log \frac{r}{r^2 - |z|^2} = 4 \left(\frac{r}{r^2 - |z|^2} \right)^2, \; |z| < r.$$

proof of Ahlfors' lemma

It is easy, using the chain rule, to verify that $\varphi \in N\mathfrak{C}(\tilde{\Omega})$ implies $f^*\varphi = |f'|(\varphi \circ f)$ is in $N\mathfrak{C}(\Omega)$ if $f : \Omega \to \tilde{\Omega}$ is a holomorphic maps of open connected sets in $\mathbb{C}$.

Fix $\zeta \in \mathbb{D}$, and let $r \in (|\zeta|, 1)$. Put $p_r(z) = \frac{r}{r^2 - |z|^2}$ on $B(0,r)$. Since $p_r(z) \to \infty$ as $|z| \to r$ and $f^*\varphi$ is continuous on $B[0,r]$, it is clear that the function $\psi := \frac{f^*\varphi}{p_r}$ attain its maximum on $B(0,r)$ at some $\xi \in B(0,r)$. If $(f^*\varphi)(\xi) = 0$, then $\varphi \equiv 0$.

Hence we may assume that $\xi \in D_\varphi$. Then ξ is also a local maximum of $\log\psi$, and it follows that $\Delta \log \psi \leq 0$ at q. Now, at ξ:

$$\begin{aligned} 0 \geq \Delta \log \psi &= \Delta \log f^*\varphi - \Delta \log p_r \\ &\geq 4(f^*\varphi^2 - p_r^2), \end{aligned}$$

that is, $\psi(\xi) \leq 1$. Thus $f^*\varphi \leq p_r$ on $B(0,r)$. Letting $r \uparrow 1$, we conclude that $(f^*\varphi)(z) \leq \frac{1}{1-|p|^2}$, $z \in B(0,1)$, as required.

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applications

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Definition: For any open connected set $\Omega \subseteq \mathbb{C}$, let $N\mathfrak{C}(\Omega)$ denote the set of continuous functions $\varphi \geq 0$ on Ω such that φ is C^2 on $\{\varphi > 0\}$ and $\Delta \log \varphi \geq 4\varphi^2$ there.

As we have said before, using the chain rule, it is easy to verify:

Proposition: Suppose that $f : \Omega \to \tilde{\Omega}$ is a holomorphic maps of open connected sets in $\mathbb{C}$. Then $\varphi \in N\mathfrak{C}(\tilde{\Omega})$ implies $f^*\varphi = |f'|(\varphi \circ f)$ is in $N\mathfrak{C}(\Omega)$.

It follows from the Ahlfors lemma that $N\mathfrak{C}(\mathbb{C}) = \{0\}$.

Verification: Pick φ in $N\mathfrak{C}(\mathbb{C})$. Fix $a \in \mathbb{C}$. For any $r > |a|$, taking $f : B(0,r) \to B(0,r)$, $f(z) = z$, Ahlfors Lemma yields $(f^*\varphi)(a) = \varphi(a) \leq \frac{r}{r^2 - |a|^2}$. As $r \to \infty$, we see that $\varphi(a) = 0$.

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Corollary: Let $f : \mathbb{C} \to \Omega$, $\Omega \subseteq \mathbb{C}$, be a holomophic function. If $N\mathfrak{C}(\Omega) \neq \{0\}$, then f must be constant.

Liouville's theorem: As a corollary, taking $\Omega = B(0, M)$, we see that every bounded entire function must be a constant.

Picard's little theorem: Similarly, if $f : \mathbb{C} \to \mathbb{C}_{\{0,1\}}$, where $\mathbb{C}_{\{0,1\}} := \mathbb{C} \backslash \{0, 1\}$ is holomophic, then f is constant.

proof: To verify this, all we need to do is show that $N\mathfrak{C}(\mathbb{C}_{\{0,1\}}) \neq \{0\}$. The non-zero function

$$\varphi(z) = |z|^{\beta/2-1}|1-z|^{\beta/2-1}(1+|z|^{\beta})(1+|z-1|)^{\beta},\ \beta > 0,$$

is in $N\mathfrak{C}(\mathbb{C}_{\{0,1\}})$ for $0 < \beta < 2/7$.

Corollary: Let $f:\mathbb{C}\to\Omega$, $\Omega\subseteq\mathbb{C}$, be a holomophic function. If $N\mathfrak{C}(\Omega)\neq\{0\}$, then f must be constant.

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Thank You!